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A theory of time-dependent, isotropic turbulence

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Abstract. In this paper second-order equations are derived for the turbulent velocity-field correlation and propagator functions. It is argued that the concept of the propagator may be more fully exploited as the relationship between eddies at successive times, than as the relationship between the velocity and the arbitrary stirring forces. The resulting equations differ from the direct-interaction approximation of Kraichnan by the presence of additional diffusive-type terms in the equation for the propagator. A generalisation of the diagram technique due to Wyld is used to analyse the approximation procedure to fourth order. It is shown that many higher-order terms in the perturbation series are represented in the truncated equations.

As a first step in assessing these equations, steady-state forms are obtained, on the assumption of exponential time dependences. The resulting equation for the effective viscosity is found to be similar to a previous theory of McComb. But there are important differences: the integrand is not antisymmetric and there is no cut-off in wavenumber. The time-independent form of the theory is found to be compatible with the Kolmogoroff distribution. It is concluded that this is sufficiently encouraging to justify a numerical study of the time-dependent equations.

1. Introduction

Recent attempts to develop a fundamental theory of turbulence may be divided into two types of approach. The first of these was initiated by Kraichnan (1959, and many other papers: see the book by Leslie 1973 for a full account), who introduced a new perturbation method for solving the Navier–Stokes equations. Kraichnan derived a set of coupled integral equations, for the velocity correlation and response functions, which took into account the collective nature of turbulent motion. This kind of direct manipulation of the equations of motion has also been the basis of subsequent theories, including Wyld (1961), Lee (1965), Pythian (1969) and Nakano (1972).

The second approach stems from the work of Hopf (1952), who used the equation of motion to derive an analogue of the Liouville equation, for the distribution function of the fluctuating velocities. Edwards (1964) took the Liouville equation as a starting point and obtained closed equations for the energy spectrum and dynamical friction (or effective viscosity). With appropriate assumptions about the form of time dependences, these results were found to be compatible with those of Kraichnan. Examples of further work in this area include papers by Herring (1965), Edwards and McComb (1969) and Balescu and Senatorski (1970).

The two types of theory have in common the development of modified perturbation methods to handle what is a strongly non-linear problem. Another common feature is the recognition that a second function, be it response function, velocity propagator or effective viscosity, must be introduced in order to obtain a closed set of equations for the energy spectrum. Variations between theories tend to be in terms of the way this second function is introduced (and in the final form of its governing equations) as all approaches seem to lead to similar equations for the energy spectrum.

So far the expansion methods that have been used are not well justified or understood. Many questions remain unanswered about convergence, uniqueness of solutions and the importance of terms which have been neglected. Given the very complicated nature of the higher-order terms, this is not surprising; and at present the main emphasis is on finding tractable forms which may be tested against experimental results. The accepted test seems to be that the theory in question should yield the well known distribution which was predicted on dimensional grounds by Kolmogoroff (see e.g. Leslie 1973).

Of the theories mentioned, only two yield the Kolmogoroff inertial-range spectrum as a solution. The first of these was due to Kraichnan (1965), who extended his direct-interaction approximation to a mixed Eulerian–Lagrangian frame of reference. In the second, Edwards and McComb (1969) chose the response function such that the turbulent entropy (defined as information content) was maximised. Both these theories are very complicated and it is also fair to say that both possess some unsatisfactory features.

In later work, Kraichnan (1971) has developed a new approach in which some of the desirable features of his 1965 theory are retained in a purely Eulerian framework. The origins of this work lie in an 'almost-Markoffian' equation, derived from a model representation based on the work by Pythian (1969). The resulting 'test-field' model yields the Kolmogoroff distribution, but does not involve the level of complication found in the earlier Lagrangian-type theory. Interestingly, the final equations are very close to those of Edwards (1964) but slight differences in the inertial-transfer operators are sufficient to remove the well known divergence from the equation for the response function.

A different approach has been followed in more recent papers (McComb 1974, 1976), where we have shown that the energy equation due to Edwards (1964) may be re-interpreted in terms of a Heisenberg-type effective viscosity. Symmetries in the nonlinear kernels were exploited to obtain a response integral, which did not diverge at the origin and hence was compatible with the Kolmogoroff distribution. Although this improved behaviour was due to the presence of additional diffusive-type terms in the kernel, a cut-off in wavenumber was also involved. This work was restricted to steady, isotropic turbulence.

In this paper we present a new theory of time-dependent isotropic turbulence in an incompressible fluid. Our approach belongs to the first type mentioned at the beginning of this section, in that we manipulate the Navier–Stokes equation directly. The additional quantity needed for closure is introduced formally as the velocity-field propagator which relates the velocity of a given eddy to its values at successive times. The equation of motion for this quantity is obtained from a generalisation of the equation for the velocity covariance. In this respect the basic idea is similar to our recent steady-state theory (McComb 1974, 1976), where the response function was determined from the energy balance in wavenumber space. But it will be seen that a completely different analytical path is followed and that the symmetry arguments (and associated wavenumber cut-offs) used previously, do not occur in the present theory.

It will also be seen that (unlike some other approaches) we do not introduce a formal relationship between the velocity field and an arbitrary stirring force. This does not rule out such forces. Clearly one may postulate that random stirring forces give rise to an energy input which will sustain the turbulence (Edwards and McComb 1969). Thus there is no inconsistency in (for example) considering the turbulent field to be stationary and, in fact, we shall find it convenient to do so.

2. The basic equations

In this section we summarise those aspects of current methods of treating the Navier–Stokes equation which will be needed in this paper. We begin by considering an incompressible fluid in a cubical box of side L. At a later stage we take the limit $L \to \infty$ (which is required for rigorous isotropy) and summations are replaced by integrals. If we let the velocity field be $U_\alpha(\boldsymbol{x}, t)$ then the Fourier components of this are defined by

$$U_\alpha(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} U_\alpha(\boldsymbol{k}, t)\, e^{i\boldsymbol{k}.\boldsymbol{x}}. \tag{2.1}$$

For an incompressible fluid, the continuity equation becomes

$$k_\alpha U_\alpha(\boldsymbol{k}, t) = 0 \tag{2.2}$$

and the Navier–Stokes equation may be written

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U_\alpha(\boldsymbol{k}, t) = \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\beta(\boldsymbol{j}, t) U_\gamma(\boldsymbol{k}+\boldsymbol{j}, t) \tag{2.3}$$

where the inertial-transfer operator $M_{\alpha\beta\gamma}(\boldsymbol{k})$ is defined by

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = \tfrac{1}{2}\mathrm{i}(k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})) \tag{2.4}$$

and

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\boldsymbol{k}|^{-2} \tag{2.5}$$

(e.g. see the book by Leslie 1973).

The pair correlation of velocities may be defined, thus:

$$\left(\frac{L}{2\pi}\right)^3 \langle U_\alpha(\boldsymbol{k}, t) U_\beta(-\boldsymbol{k}, t')\rangle = Q_{\alpha\beta}(\boldsymbol{k}, t-t') \tag{2.6}$$

where $\langle\ \rangle$ means average value, and the turbulence is assumed to be stationary in time. If the turbulence is isotropic we may further write

$$Q_{\alpha\beta}(\boldsymbol{k}, t-t') = D_{\alpha\beta}(\boldsymbol{k}) Q(k, t-t'). \tag{2.7}$$

As $k_\alpha D_{\alpha\beta}(\boldsymbol{k}) = 0$, the continuity equation will be satisfied by (2.7), for an arbitrary scalar function $Q(k, t-t')$, which depends only on the magnitude of the wavevector $\boldsymbol{k}$.

In order to obtain an iterative solution of equation (2.3) we make the expansion

$$U_\alpha(\boldsymbol{k}, t) = U_\alpha^{(0)}(\boldsymbol{k}, t) + \lambda U_\alpha^{(1)}(\boldsymbol{k}, t) + \lambda^2 U_\alpha^{(2)}(\boldsymbol{k}, t) + \mathrm{O}(\lambda^3). \tag{2.8}$$

Here λ is an ordering parameter and will be put equal to unity at the end of the calculation. Superficially λ is of order $M_{\alpha\beta\gamma}(\boldsymbol{k})$ as, when any coefficient in (2.8) is expressed wholly in terms of the $U^{(0)}$, then the associated power of λ is equal to the number of inertial-transfer operators occurring in the expression. The zero-order velocity field $U^{(0)}$ is taken to be prescribed and to have Gaussian statistics. While it is possible to add a stirring force to the right-hand side of (2.3) to sustain the turbulence, and then to relate $U^{(0)}$ to the stirring force, we shall not make use of that approach here.

The iteration is based on the inversion of the linear operator on the right-hand side of (2.3) so it is convenient to introduce the zero-order propagator $H^{(0)}_{\alpha\beta}(\boldsymbol{k}, t-t')$, thus:

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H^{(0)}_{\alpha\beta}(\boldsymbol{k}, t-t') = D_{\alpha\beta}(\boldsymbol{k})\delta(t-t') \tag{2.9}$$

and hence

$$H^{(0)}_{\alpha\beta}(\boldsymbol{k}, t-t') = \begin{cases} D_{\alpha\beta}(\boldsymbol{k})\exp[-\nu k^2(t-t')] & t>t' \\ 0 & t<t'. \end{cases} \tag{2.10}$$

Then, substituting the expansion (2.8) into equation (2.3) and associating a factor λ with $M_{\alpha\beta\gamma}(\boldsymbol{k})$, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\nu k^2\right)U_\alpha(\boldsymbol{k}, t) \\ &= \lambda\sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})(U^{(0)}_\beta(\boldsymbol{j}, t)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)+\lambda U^{(0)}_\beta(\boldsymbol{j}, t)U^{(1)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t) \\ &\quad +\lambda U^{(1)}_\beta(\boldsymbol{j}, t)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)+\ldots.\end{aligned} \tag{2.11}$$

The coefficients in (2.8) are obtained by equating powers of λ, and using (2.10) to obtain:

$$\begin{aligned} U^{(1)}_\alpha(\boldsymbol{k}, t) &= \int_{-\infty}^{t} \mathrm{d}s_1 H^{(0)}_{\alpha\alpha'}(\boldsymbol{k}, t-s_1)\sum_{\boldsymbol{j}_1} M_{\alpha'\beta\gamma}(\boldsymbol{k})U^{(0)}_\beta(\boldsymbol{j}_1, s_1)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}_1, s_1) \\ U^{(2)}_\alpha(\boldsymbol{k}, t) &= \int_{-\infty}^{t} \mathrm{d}s\, H^{(0)}_{\alpha\alpha'}(\boldsymbol{k}, t-s_1)\sum_{\boldsymbol{j}_1} M_{\alpha'\beta\gamma}(\boldsymbol{k})(U^{(0)}_\beta(\boldsymbol{j}_1, s_1)U^{(1)}_\gamma(\boldsymbol{k}+\boldsymbol{j}_1, s_1) \\ &\quad + U^{(1)}_\beta(\boldsymbol{j}_1, s_1)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}_1, s_1)) \\ &\vdots \qquad\qquad \vdots \end{aligned} \tag{2.12}$$

And, by repeated substitution:

$$\begin{aligned} U^{(1)}_\alpha(\boldsymbol{k}, t) &= \int_{-\infty}^{t} \mathrm{d}s_1 H^{(0)}_{\alpha\alpha'}(\boldsymbol{k}, t-s_1)\sum_{\boldsymbol{j}_1} M_{\alpha'\beta\gamma}(\boldsymbol{k})U^{(0)}_\beta(\boldsymbol{j}_1, s_1)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}_1, s_1) \\ U^{(2)}_\alpha(\boldsymbol{k}, t) &= 2\int_{-\infty}^{t} \mathrm{d}s_1 \int_{-\infty}^{s_1} \mathrm{d}s_2 \sum_{\boldsymbol{j}_1,\boldsymbol{j}_2} H^{(0)}_{\alpha\alpha'}(\boldsymbol{k}, t-s_1)M_{\alpha'\beta\gamma}(\boldsymbol{k})U^{(0)}_\beta(\boldsymbol{j}_1, s_1) \\ &\quad \times(H^{(0)}_{\gamma'\gamma''}(\boldsymbol{k}+\boldsymbol{j}_1, s_1-s_2)M_{\gamma''\beta'\gamma}(\boldsymbol{k}+\boldsymbol{j}_1)U^{(0)}_{\beta'}(\boldsymbol{j}_2, s_2)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}_1+\boldsymbol{j}_2, s_2)) \\ &\vdots \qquad\qquad \vdots \end{aligned} \tag{2.13}$$

Higher-order terms are more conveniently expressed in terms of diagrams (Wyld 1961, Lee 1965). We will make use of this technique in a later section.

3. The velocity-field propagator

The idea that closure of the main Navier–Stokes hierarchy should be in terms of the velocity correlation and propagator functions is already present in the work of Wyld (1961) and Lee (1965). These authors have also defined the exact propagator in terms of an expansion for the pair correlation rather than (say) for the turbulent velocity field (e.g. Nakano 1972). However, their formulation of the theory is aimed at a relationship between the *exact* turbulent velocity and the arbitrary stirring forces *at all times*. To us, this does not seem to fully exploit the notion of a propagator, which relates the velocity field (of a particular eddy) to itself at successive times. Also, we share some of the reservations expressed by Balescu and Senatorski (1970) about giving the arbitrary stirring forces such a prominent role in the theory.

In the present paper we introduce the propagator formally in terms of the time dependence of the velocity field. We then derive its equation of motion from the Navier–Stokes equation. This procedure is closer in spirit to the approach of Kraichnan (although the definition of the propagator or response function is quite different) than to that of Wyld, who classified diagrams in the perturbation series into propagator-like and vertex-like terms, on a topological basis.

Some preliminary discussion should help to make our approach clear, so let us consider how the concept of an exact propagator arises. The solution of equation (2.3) may be written, using (2.9) and (2.10), as

$$U_\alpha(\boldsymbol{k},t)=H^{(0)}_{\alpha\sigma}(\boldsymbol{k},t-t_0)U_\sigma(\boldsymbol{k},t_0)+\int_{t_0}^{t}\mathrm{d}s H^{(0)}_{\alpha\sigma}(\boldsymbol{k},t-s)\sum_{\boldsymbol{j}}M_{\sigma\beta\gamma}(\boldsymbol{k})U_\beta(\boldsymbol{j},s)U_\gamma(\boldsymbol{k}+\boldsymbol{j},s). \tag{3.1}$$

The role of $H^{(0)}$ as a zero-order propagator is clear. It satisfies

$$U^{(0)}_\alpha(\boldsymbol{k},t)=H^{(0)}_{\alpha\sigma}(\boldsymbol{k},t-s)U^{(0)}_\sigma(\boldsymbol{k},s) \tag{3.2}$$

and has the properties

$$\begin{aligned}&H^{(0)}_{\alpha\sigma}(\boldsymbol{k},t-s)H^{(0)}_{\sigma\beta}(\boldsymbol{k},s-t')=H^{(0)}_{\alpha\beta}(\boldsymbol{k},t-t')\\&H^{(0)}_{\alpha\sigma}(\boldsymbol{k};t,t)=1.\end{aligned} \tag{3.3}$$

Now consider equation (3.1) with t_0 taken sufficiently far in the past for the first term on the right-hand side to be neglected, thus:

$$U_\alpha(\boldsymbol{k},t)=\int_{-\infty}^{t}\mathrm{d}s\, H^{(0)}_{\alpha\sigma}(\boldsymbol{k},t-s)\sum_{\boldsymbol{j}}M_{\sigma\beta\gamma}(\boldsymbol{k})U_\beta(\boldsymbol{j},s)U_\gamma(\boldsymbol{k}+\boldsymbol{j},s). \tag{3.4}$$

Clearly this is the exact solution for $U_\alpha(\boldsymbol{k},t)$ but the unknown interaction term makes it of little use. The introduction of an exact propagator H is equivalent to saying that, by analogy with (3.2) and (3.3), equation (3.4) can be approximated by

$$U_\alpha(\boldsymbol{k},t)=H_{\alpha\sigma}(\boldsymbol{k},t-s)U_\sigma(\boldsymbol{k},s) \tag{3.5†}$$

† It should be noted that H, as defined by equation (3.5), is statistically sharp. Thus H may be treated as independent of any particular realisation of the velocity field. The use of (3.5) may be regarded as somewhat phenomenological but we would argue that the term 'exact propagator' is not inappropriate because of the dependence of H on the *moments* of the turbulent velocity field.

where

$$H_{\alpha\sigma}(\boldsymbol{k}, t-s)H_{\sigma\beta}(\boldsymbol{k}, s-t') = H_{\alpha\beta}(\boldsymbol{k}, t-t')$$
$$H_{\alpha\sigma}(\boldsymbol{k}; t, t) = 1. \tag{3.6}$$

Thus, by making the usual analogy between the behaviour of the linear and non-linear operators, we are essentially forcing equation (3.4) into the form of (3.5). The implications of this step are clearer if we multiply both sides of (3.4) by $U_{\alpha'}(-\boldsymbol{k}, t')$, and average, to obtain

$$Q_{\alpha\alpha'}(\boldsymbol{k}, t-t') = \int_{-\infty}^{t} \mathrm{d}s\, H^{(0)}_{\alpha\sigma}(\boldsymbol{k}, t-s)\left(\frac{L}{2\pi}\right)^3 \sum_{j} M_{\sigma\beta\gamma}(\boldsymbol{k})\langle U_\beta(\boldsymbol{j}, s)U_\gamma(\boldsymbol{k}+\boldsymbol{j}, s)U_{\alpha'}(-\boldsymbol{k}, t')\rangle. \tag{3.7}$$

In general we would not expect this to reduce to the simple form appropriate to equation (3.5) but to something rather more complicated like (say)

$$Q_{\alpha\alpha'}(\boldsymbol{k}, t-t') = A_{\alpha\sigma}(\boldsymbol{k}, t-s)Q_{\alpha\sigma'}(\boldsymbol{k}, s-t') + \sum_{j} B_{\alpha\sigma}(\boldsymbol{k}+\boldsymbol{j}, t-s)Q_{\sigma\alpha'}(\boldsymbol{j}, s-t') \tag{3.8}$$

where A and B are (at the very least) functionals of the pair correlation Q's. Indeed, existing energy-balance equations (e.g. Kraichnan 1959, Edwards 1964) strongly suggest that (3.8) represents the minimum level of complexity needed to approximate equation (3.7) for the pair correlation. Hence, in order to obtain a propagator as defined by equation (3.5), one must either neglect the second (diffusive-type) term on the right of (3.8) or alternatively force (3.8) into the appropriate form in some way.

For example, temporarily suppressing tensor indices, we might write equation (3.8) as

$$Q(\boldsymbol{k}, t-t') = \left(A(\boldsymbol{k}, t-s) + \sum_{j} Q^{-1}(\boldsymbol{k}, s-t')B(\boldsymbol{k}+\boldsymbol{j}, t-s)Q(\boldsymbol{j}, s-t')\right)Q(\boldsymbol{k}, s-t') \tag{3.9}$$

which has the required form. Of course this is a clumsy manoeuvre and is only introduced to make our point; that diffusive-type terms can be retained in the equation for the propagator. In practice it would not be very helpful.

We conclude this section by noting that in most of the theories mentioned earlier, the equation for the response function or propagator only involves the first type of term on the right-hand side of equation (3.8). A particularly interesting example is provided by the work of Nakano (1972) who used an iterative technique to reduce equation (3.5) to (3.4), with H in the form of an expansion. Nakano divided the diagrammatic representation of H into two classes on a topological basis and neglected one class of terms in order to obtain a series that could be summed. But the terms which were neglected are just those which would contribute to a diffusive-type second term as in (3.8).

Exceptions to the above are the entropy method (Edwards and McComb 1969) and the local energy transfer theory (McComb 1974, 1976). In both of these, diffusive-type terms arise naturally in the equation for the response (or strictly, the effective viscosity). However, both theories are for steady turbulence, and there is an underlying assumption of exponential time dependence. Thus the problems inherent in the inversion of the pair-correlation (e.g. as in (3.9)) are smoothed out. In the next section we shall show that it is also possible to cope with this problem in the general time-dependent case.

4. Equations for the correlation and propagator functions

We begin by deriving a generalised equation for the pair correlation. Using the definition of the exact propagator, equation (3.5), we write the Navier–Stokes equation as

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H_{\alpha\sigma}(\boldsymbol{k}, t-s)U_\sigma(\boldsymbol{k}, s)=\lambda \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})U_\beta(\boldsymbol{j}, t)U_\gamma(\boldsymbol{k}+\boldsymbol{j}, t) \quad (4.1)$$

where λ is the ordering parameter. Now multiply both sides of (4.1) by

$$U_{\alpha'}(-k, t')=H_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')U_{\sigma'}(-\boldsymbol{k}, s') \quad (4.2)$$

and average, to obtain

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H_{\alpha\sigma}(\boldsymbol{k}, t-s)\langle U_\sigma(\boldsymbol{k}, s)U_{\sigma'}(-\boldsymbol{k}, s)\rangle H_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')$$

$$=\lambda \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})H_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')\langle U_\beta(\boldsymbol{j}, t)U_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U_{\sigma'}(-\boldsymbol{k}, s)\rangle. \quad (4.3)$$

In order to expand the right-hand side of (4.3), we use (2.8) for the velocity field and, for the propagator, we introduce the corresponding expansion

$$H_{\alpha\beta}(\boldsymbol{k}, t-s)=H^{(0)}_{\alpha\beta}(\boldsymbol{k}, t-s)+\lambda^2 H^{(2)}_{\alpha\beta}(\boldsymbol{k}, t-s)+\mathrm{O}(\lambda^4). \quad (4.4)$$

Also it is convenient at this stage to note that isotropy implies the form

$$H_{\alpha\beta}(\boldsymbol{k}, t-s)=D_{\alpha\beta}(\boldsymbol{k})H(k, t-s) \quad (4.5)$$

where $D_{\alpha\beta}(\boldsymbol{k})$ is given by equation (2.5). As in (2.7) for $Q_{\alpha\beta}(\boldsymbol{k}, t-t')$, the projection operator ensures that the incompressibility condition is satisfied for an arbitrary function $H(k, t-s)$, which depends only on the scalar magnitude of $\boldsymbol{k}$.

Substituting from (2.8) and (4.4), equation (4.3) becomes

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H_{\alpha\sigma}(\boldsymbol{k}, t-s)\langle U_\sigma(\boldsymbol{k}, s)U_{\sigma'}(-\boldsymbol{k}, s')\rangle H_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')$$

$$\begin{aligned}=\lambda \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})&[H^{(0)}_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')+\lambda^2 H^{(2)}_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')+\mathrm{O}(\lambda^4)]\\ &\times[\langle U^{(0)}_\beta(\boldsymbol{j}, t)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U^{(0)}_{\sigma'}(-\boldsymbol{k}, s')\rangle\\ &+\lambda\langle U^{(0)}_\beta(\boldsymbol{j}, t)\times U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U^{(1)}_{\sigma'}(-\boldsymbol{k}, s')\rangle\\ &+2\lambda\langle U^{(1)}_\beta(\boldsymbol{j}, t)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U^{(0)}_{\sigma'}(-\boldsymbol{k}, s')\rangle+\mathrm{O}(\lambda^2)]. \quad (4.6)\end{aligned}$$

We shall only work to second order in this analysis but, in the next section, diagrammatic methods will be used to investigate the higher-order structure. In this connection it will be important to note that the term in $H^{(2)}$ will also contribute to the fourth order.

Substituting from (2.12) for $U^{(1)}$ we then obtain

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H_{\alpha\sigma}(\boldsymbol{k}, t-s)\langle U_\sigma(\boldsymbol{k}, s)U_{\sigma'}(-\boldsymbol{k}, s')\rangle H_{\alpha'\sigma'}(\boldsymbol{k}, s'-t)$$

$$\begin{aligned}
&=\sum_j M_{\alpha\beta\gamma}(\boldsymbol{k})H^{(0)}_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')\Big(\lambda\langle U^{(0)}_\beta(\boldsymbol{j}, t)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U^{(0)}_{\sigma'}(-\boldsymbol{k}, s')\rangle\\
&+\lambda^2\sum_{j'}\int_{-\infty}^{t'}\mathrm{d}s\, H^{(0)}_{\sigma'\rho}(\boldsymbol{k}, s'-s)M_{\rho\beta'\gamma'}(-\boldsymbol{k})\\
&\times\langle U^{(0)}_\beta(\boldsymbol{j}, t)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U^{(0)}_{\beta'}(\boldsymbol{j}', s)U^{(0)}_{\gamma'}(-\boldsymbol{k}+\boldsymbol{j}', s)\rangle\\
&+2\lambda^2\sum_{j'}\int_{-\infty}^{t}\mathrm{d}s\, H^{(0)}_{\beta\rho}(\boldsymbol{j}, t-s)M_{\rho\beta'\gamma'}(\boldsymbol{j})\\
&\times\langle U^{(0)}_{\beta'}(\boldsymbol{j}', s)U^{(0)}_{\gamma'}(\boldsymbol{j}+\boldsymbol{j}', s)U^{(0)}_\gamma(\boldsymbol{k}+\boldsymbol{j}, t)U^{(0)}_{\sigma'}(-\boldsymbol{k}, s')\rangle\Big)+\mathrm{O}(\lambda^3).
\end{aligned}\tag{4.7}$$

The $U^{(0)}$ are taken to have Gaussian statistics, so all odd-order moments vanish. The fourth-order moment may be evaluated in terms of the second-order moments, in the usual way, and (4.7) reduces to

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H_{\alpha\sigma}(\boldsymbol{k}, t-s)Q_{\sigma\sigma'}(\boldsymbol{k}, s-s')H_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')$$

$$\begin{aligned}
&=\lambda^2\left(\frac{L}{2\pi}\right)^2\sum_j\Big(2M_{\alpha\beta\gamma}(\boldsymbol{k})M_{\rho\beta'\gamma'}(-\boldsymbol{k})\int_{-\infty}^{t'}\mathrm{d}s\, H^{(0)}_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')H^{(0)}_{\alpha'\rho}(\boldsymbol{k}, s'-s)\\
&\times Q^{(0)}_{\beta\beta'}(\boldsymbol{j}, t-s)Q^{(0)}_{\gamma\gamma'}(\boldsymbol{k}+\boldsymbol{j}, t-s)\\
&+4M_{\alpha\beta\gamma}(\boldsymbol{k})M_{\rho\beta'\gamma'}(\boldsymbol{j})\int_{-\infty}^{t}\mathrm{d}s\, H^{(0)}_{\alpha'\sigma'}(\boldsymbol{k}, s'-t')H^{(0)}_{\beta\rho}(\boldsymbol{j}, t-s)\\
&\times Q^{(0)}_{\beta'\sigma'}(\boldsymbol{k}, s-s')Q^{(0)}_{\gamma\gamma'}(\boldsymbol{k}+\boldsymbol{j}, s-s)\Big)+\mathrm{O}(\lambda^4).
\end{aligned}\tag{4.8}$$

At this stage we put $\lambda=1$ and replace the zero-order correlations and propagators by their exact forms on the right-hand side of equation (4.8). We shall show in the next section that this corresponds to summing certain classes of terms in the perturbation series to all orders. We also express the correlation and response tensors in terms of the appropriate scalar functions using (2.7) and (4.15). Then, making the transition to the infinite system, and using the property of the projection operators that $D_{\alpha\alpha'}(\boldsymbol{k})D_{\alpha'\beta}(\boldsymbol{k})=D_{\alpha\beta}(\boldsymbol{k})$, we obtain from (4.8)

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)D_{\alpha\alpha'}(\boldsymbol{k})H(k, t-s)Q(k, s-s')H(k, s'-t')$$

$$\begin{aligned}
&=\int\mathrm{d}^3j\Big(2M_{\alpha\beta\gamma}(\boldsymbol{k})M_{\rho\beta'\gamma'}(-\boldsymbol{k})D_{\alpha'\rho}(\boldsymbol{k})D_{\beta\beta'}(\boldsymbol{j})D_{\gamma\gamma'}(\boldsymbol{k}+\boldsymbol{j})\\
&\times\int_{-\infty}^{t'}\mathrm{d}s\, H(k, s'-t')H(k, s'-s)Q(j, t-s)Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s)\\
&+4M_{\alpha\beta\gamma}(\boldsymbol{k})M_{\rho\beta'\gamma'}(\boldsymbol{j})D_{\alpha'\beta'}(\boldsymbol{k})D_{\beta\rho}(\boldsymbol{j})D_{\gamma\gamma'}(\boldsymbol{k}+\boldsymbol{j})\\
&\times\int_{-\infty}^{t}\mathrm{d}s\, H(k, s'-t')H(j, t-s)Q(k, s-s')Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s)\Big).
\end{aligned}\tag{4.9}$$

Finally, taking $\alpha=\alpha'$, this equation may be written with the right-hand side in a familiar form, as

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H(k,t-s)Q(k,s-s')H(k,s'-t')$$
$$=\int \mathrm{d}^3j\,L_{kj}\left(\int_{-\infty}^{t'}\mathrm{d}s\,H(k,t'-s)Q(j,t-s)Q(|\boldsymbol{k}+\boldsymbol{j}|,t-s)\right.$$
$$\left.-\int_{-\infty}^{t}\mathrm{d}s\,H(k,s'-t')H(j,t-s)Q(k,s-s')Q(|\boldsymbol{k}+\boldsymbol{j}|,t-s)\right) \tag{4.10}$$

where we have contracted the time arguments in the first term on the right to eliminate one of the propagators; and L_{kj} is given by

$$L_{kj}=\frac{(k^2j^2+2k^2j^2\mu^2+k^3j\mu+kj^3\mu)(1-\mu^2)}{k^2+2kj+j^2} \tag{4.11}$$

μ being the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{j}$. From equation (4.10) we may obtain equations for the propagator and correlation functions.

4.1. *The equation for the propagator*

We obtain this by evaluating the equation for the pair correlation on the time diagonal. Putting $s=s'$ in both sides of equation (4.10) yields:

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H(k,t-s)H(k,s-t')Q(k,0)$$
$$=\int \mathrm{d}^3j\,L_{kj}\int_{-\infty}^{t'}\mathrm{d}s\,H(k,t'-s)Q(j,t-s)Q(|\boldsymbol{k}+\boldsymbol{j}|,t-s)$$
$$-\int_{-\infty}^{t}\mathrm{d}s\,H(k,s-t')H(k,t-s)Q(k,0)Q(|\boldsymbol{k}+\boldsymbol{j}|,t-s). \tag{4.12}$$

Now we may contract time arguments on the left-hand side, according to (3.6), and divide across by $Q(k,0)$. Writing

$$Q(k,0)\equiv q_k=\frac{E(k)}{4\pi k^2} \tag{4.13}$$

where $E(k)$ is the steady-state energy spectrum we obtain

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H(k,t-t')$$
$$=\int \mathrm{d}^3j\,L_{kj}\left(q_k^{-1}\int_{-\infty}^{t'}\mathrm{d}s\,H(k,t'-s)Q(j,t-s)Q(|\boldsymbol{k}+\boldsymbol{j}|,t-s)\right.$$
$$\left.-\int_{-\infty}^{t}\mathrm{d}s\,H(k,s-t')H(j,t-s)Q(|\boldsymbol{k}+\boldsymbol{j}|,t-s)\right). \tag{4.14}$$

This is the required equation for the propagator. At this stage, perhaps the most interesting comparison is with the original direct-interaction equation for the response function (Kraichnan 1959: or see equation (6.22) in the book by Leslie 1973). Re-arranging (4.14) in the form

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\nu k^2\right)&H(k, t-t') \\
&= -\int \mathrm{d}^3 j\, L_{kj} \int_{-\infty}^{t} \mathrm{d}s\, H(j, t-s) Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s) H(k, s-t') \\
&\quad + \int \mathrm{d}^3 j\, L_{kj} \int_{-\infty}^{t'} \mathrm{d}s\, H(k, t'-s) \frac{Q(j, t-s) Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s)}{q_k}
\end{aligned}
\tag{4.15}
$$

which may be compared with Kraichnan's equation for the response function $G(k, t-t')$, which is

$$
\left(\frac{\partial}{\partial t}+\nu k^2\right) G(k, t-t') = -\int \mathrm{d}^3 j\, L_{kj} \int_{-\infty}^{t} \mathrm{d}s\, G(j, t-s) Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s) G(k, s-t'),
$$

when rewritten in our present notation. Clearly equation (4.15) differs only from Kraichnan's result by the presence of the second, diffusive-type, term on the right-hand side. It will be seen later that this term prevents the well known divergence at $k=0$, when the Kolmogoroff distributions are substituted for Q and H.

4.2. *The equation for the correlation function*

In this case, we simply make use of the property (3.5) to contract time arguments on both sides of equation (4.10), which then reduces to

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\nu k^2\right)&Q(k, t-t') \\
&= \int \mathrm{d}^3 j\, L_{kj} \int_{-\infty}^{t'} \mathrm{d}s\, H(k, t'-s) Q(j, t-s) Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s) \\
&\quad - \int \mathrm{d}^3 j\, L_{kj} \int_{-\infty}^{t} \mathrm{d}s\, H(j, t-s) Q(k, s-t') Q(|\boldsymbol{k}+\boldsymbol{j}|, t-s).
\end{aligned}
\tag{4.16}
$$

Again, the immediate comparison is with Kraichnan's energy-balance equation. Referring to equation (6.21) of Leslie's book, we see that (notational differences aside) the two equations are identical. This is not surprising as, to second order, there is no essential difference between the two derivations.

We shall return to a detailed consideration of equations (4.14) and (4.16) at a later stage. In the next section we investigate the nature of the approximations made, by considering a diagrammatic analysis of the perturbation series.

5. Diagrammatic representation of the perturbation series

5.1. *The equation for the pair-correlation*

A general analysis of the perturbation series, using graphical methods, has been given by Wyld (1961), who considered a one-dimensional form of the equation of motion. The series expansion for the velocity field (2.8) was substituted directly into the definition of the pair correlation (2.6) and the zero-order propagator used to relate the zero-order velocity to an arbitrary stirring force. Generalised diagram parts were identified as propagator-like or vertex-like, in terms of their mode of connection to other diagram parts. Both vertex and line renormalisation were employed to reduce the set of diagrams. It was found that one class of diagrams could be summed exactly. This provided a relationship between the pair correlation of velocities and the auto-correlation of the stirring forces, at all times. The second class could be expressed as a series of irreducible diagrams. Similar expansions were obtained for the exact propagator and vertex functions. A non-trivial generalisation of this work to three-dimensional turbulence was given by Lee (1965), who also considered the hydro-magnetic case.

The formalism of Wyld is attractive, in that all diagrams are accounted for, and the complexity of the turbulence problem is reduced in an impressive way. However, the final equations are still very complicated and it is necessary to make some further reduction to obtain tractable forms. For example, the direct-interaction approximation is recovered by discarding vertex corrections, summing a subset of diagrams and truncating at second order (Wyld 1961). Also, as we pointed out earlier, this formalism is influenced by the earlier work of Kraichnan (1959) and is biased towards the concept of the stirring forces maintaining the turbulence at all times. This means that Wyld's particular prescription for classifying and summing diagrams, is not intrinsically suited to our present approach.

Thus our object in this section is not to try to develop an analysis like Wyld's, in which all loose ends are tied up. Rather, we simply wish to show: (i) which classes of diagrams are retained in our approximation; and (ii) that our method of obtaining the propagator corresponds to a plausible method of classifying and summing diagrams. To this end, we will introduce diagrams which are a generalisation of those due to Wyld. The analysis will be simplified by neglecting vertex corrections. All diagrams which contain a generalised vertex part will be omitted. This point will be explained more fully at later stage.

We also have the problem that we are treating the right-hand side of the energy equation, which means that each term contains three velocity coefficients, rather than two, as in Wyld's direct expansion for the pair correlation. It will be seen that we can deal with this by replacing two of the coefficients in each term by a term involving one coefficient of higher degree.

To set up the diagram method we first re-derive the energy equation, using a crudely symbolic notation. Wavenumber and time arguments are suppressed, along with tensor indices. Only the degree and the ordering of coefficients will be significant.

We start by writing equation (2.3) as

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)U=\lambda MUU \tag{5.1}$$

and the perturbation expansion (2.8) as

$$U=U_0+\lambda U_1+\lambda^2U_2+\lambda^3U_3+O(\lambda^4). \tag{5.2}$$

The coefficients, as given by (2.12), now become

$$\begin{aligned} U_1 &= H_0 M U_0 U_0 \\ U_2 &= 2H_0 M U_0 U_1 \\ U_3 &= 2H_0 M U_0 U_2 + H_0 M U_1 U_1 \\ U_4 &= 2H_0 M (U_0 U_3 + U_1 U_2) \\ &\vdots \end{aligned} \tag{5.3}$$

Equation (5.2) is used to expand the right-hand side of (5.1) to obtain

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U = M[\lambda U_0 U_0 + 2\lambda^2 U_0 U_1 + 3\lambda^3 U_0 U_1 + \lambda^3 U_1 U_1 + 2\lambda^4 U_0 U_3 + 2\lambda^4 U_1 U_2 + \mathrm{O}(\lambda^5)] \tag{5.4}$$

which is then multiplied on both sides by

$$HU = [H_0 + \lambda^2 H_2 + \mathrm{O}(\lambda^4)][U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3 + \mathrm{O}(\lambda^4)] \tag{5.5}$$

and the result averaged, to obtain

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu k^2\right)\langle UHU \rangle &= M[2\lambda^2 \langle U_0 U_1 H_0 U_0 \rangle + \lambda^2 \langle U_0 U_0 H_0 U_1 \rangle + \lambda^4 (2\langle U_0 U_3 H_0 U_0 \rangle \\ &\quad + 2\langle U_1 U_2 H_0 U_0 \rangle + 2\langle U_0 U_2 H_0 U_1 \rangle + \langle U_1 U_1 H_0 U_1 \rangle + 2\langle U_0 U_1 H_0 U_2 \rangle \\ &\quad + 2\langle U_0 U_0 H_0 U_3 \rangle + 2\langle U_0 U_1 H_2 U_0 \rangle + \langle U_0 U_0 H_2 U_1 \rangle) + \mathrm{O}(\lambda^6)]. \end{aligned} \tag{5.6}$$

The exact propagator H may be eliminated from the left-hand side, by contracting time arguments. Similarly, the zero-order internal propagator may be eliminated from the right-hand side by contraction with an appropriate U_0 at this, or a later stage. By using (5.3) to substitute for velocity products like $U_0 U_1$, etc, and putting $\lambda = 1$, equation (5.6) becomes:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu k^2\right)\langle U^2 \rangle &= (H_0^{-1} \langle U_2 U_0 \rangle + H_0^{-1} \langle U_1 U_1 \rangle + H_0^{-1} \langle U_4 U_0 \rangle + H_0^{-1} \langle U_3 U_1 \rangle \\ &\quad + H_0^{-1} \langle U_2 U_2 \rangle + H_0^{-1} \langle U_1 U_3 \rangle + H_0^{-1} \langle U_2 H_2 U_0 \rangle + H_0^{-1} \langle U_1 H_2 U_1 \rangle). \end{aligned} \tag{5.7}$$

We are now in a position to introduce a set of diagrams corresponding to the terms of the perturbation series, and hence to the right-hand side of equation (5.7). Let us represent the elements of the series in the following way:

full straight line $\leftrightarrow$ U_0

broken straight line $\leftrightarrow$ H_0

point (vertex) $\leftrightarrow$ M.

Then the perturbation series may be represented by diagrams as in figure 1. It should be noted that three lines meet at each vertex and that there is always wavenumber conservation at a vertex.

In order to calculate the velocity correlations, we place two diagrams, with their 'branches' facing each other, and join up emergent full lines in all possible ways. A 'cross' at the junction point indicates that two velocities are correlated. The corresponding numerical factors are obtained by multiplying the product of the factors, as given in figure 1, by the number of different ways the full lines can be joined up to form duplicate diagrams. Diagrams which correspond to a uniform translation of the system are omitted.

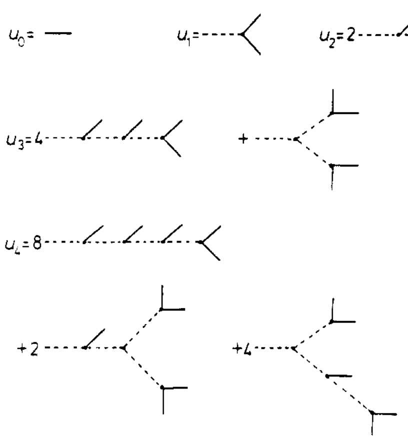

Figure 1. Diagrams corresponding to the terms of the perturbation series: equation (5.3).

A simple example will make all this clear. Let us consider the direct calculation of the pair correlation (as originally made by Wyld 1961) to second order: this is a particularly convenient example, as we shall need this result later. We have

$$Q(k)=Q_0(k)+\langle U_0U_2\rangle+\langle U_1U_1\rangle+\langle U_2U_0\rangle+\mathrm{O}(\lambda^4). \tag{5.8}$$

Joining up diagrams for $\langle U_0U_2\rangle$ etc, as explained above, results in the diagrams shown in figure 2.

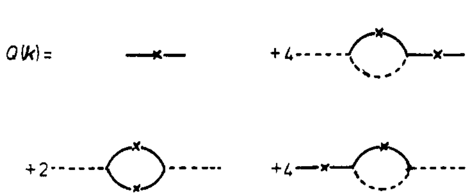

Figure 2. Expansion of the pair correlation to second order.

The diagrams corresponding to the right-hand side of equation (5.7) may be obtained in the same way. The presence of the factor H_0^{-1} in (5.7) does not present any problems. This acts from the right and cancels the emergent H_0 on the left of the diagram. We shall only work to fourth order, but it should be noted that the expansion of the propagator means that some fourth-order terms contain H_2. At this stage the diagrams corresponding to H_2 are unknown. The simplest method of dealing with this is to introduce a symbol for H_2 as a diagram part. This is then inserted in the second-order diagrams, in all permissible ways, to generate further fourth-order diagrams. The set of diagrams thus obtained is shown in figure 3. As pointed out earlier, for simplicity we have omitted all diagrams which are reducible to a lower order by replacing a diagram part by a simple vertex. For the sake of completeness, the diagrams which have been left out are listed in figure 4. Thus figures 3 and 4, together, represent the total set of diagrams, to fourth order, for the right-hand side of the energy equation.

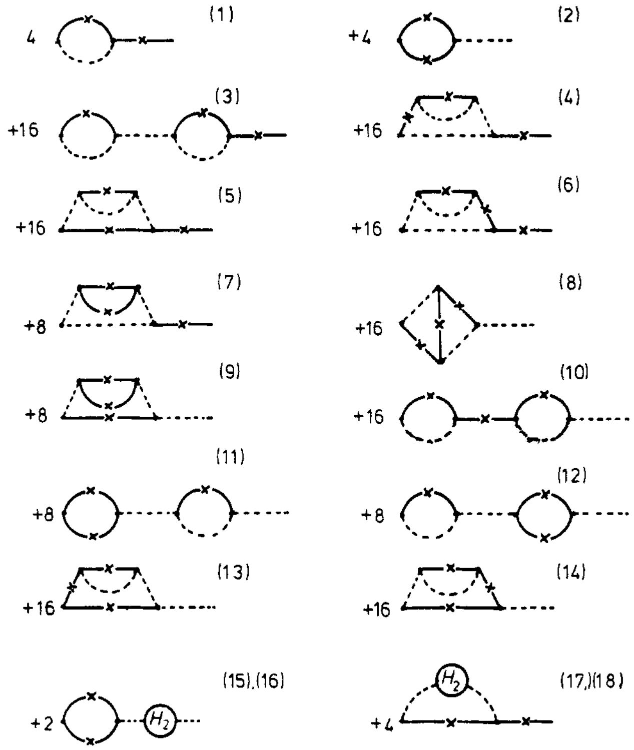

Figure 3. Diagrams corresponding to the terms on the right-hand side of equation (5.7) for the pair correlation. (Diagrams (16) and (18) are generated, along with explicit forms for (15) and (17), when the appropriate second-order diagram parts are substituted for H_2.)

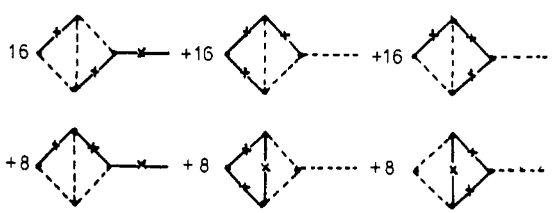

Figure 4. Fourth-order diagrams corresponding to the vertex corrections which have been omitted from figure 3.

5.2. *The expansion for the propagator*

The diagrams corresponding to the perturbation series for the propagator are generated from the set given in figure 3. The rules for this may be deduced from a comparison of the derivations of (4.14) and (4.16). We begin by noting that the diagrams of figure 3 may be divided into two classes. There are those in which a broken line emerges to the right (class I) and those in which there is an emergent full line to the right (class II). The two classes have different properties and will be considered separately.

Class I comprises diagrams (2) and (8)–(16). In deriving the equation of motion for the propagator, we divided across by a factor q_k. This must result in a factor of q_k^{-1} in class I diagrams, and a suitable generalisation has to be introduced. We may do this by associating the factor q_k^{-1} with a vertex, and representing the modified vertex by a circle.

Class II diagrams are (1), (3)–(7) and (17)–(18). These present more of a problem. The emergent full line represents $Q_0(k, t-t')$. This must be replaced by a broken line (H_0) and the diagram multiplied by $Q_0(k; 0) \equiv q_{(0)k}$. Dividing across by q_k, as in the class I diagrams, means that the diagram is multiplied by $q_k^{-1} q_{(0)k}$. Clearly q_k^{-1} should be expanded for consistency.

We may deal with this slight awkwardness by anticipating the final result. When deriving equation (4.14) we divided across by q_k after zero-order correlations had been replaced by their exact forms. Thus in the second term on the right-hand side, the factor was $q_k^{-1} q_k = 1$. If we expand q_k^{-1} on the left of a given term and q_k on the right, there is a cancellation in the product at second order. Thus, in the diagrams, $q_k^{-1} q_k = q_{(0)k}^{-1} q_{(0)k} = 1$, to fourth order, and we do not have to consider a second-order term in an expansion of q_k^{-1}.

With the above points in mind, we may write down a set of rules for obtaining the diagrams for H, from those in figure 3. These are as follows.

(a) Each class I diagram in figure 3 is multiplied by a factor q_k^{-1}. We associate this, for convenience, with the left-hand vertex. As a representation, we replace the vertex point by an open circle.

(b) In each class II diagram, the emergent full line on the right is replaced by a broken line. Each diagram is also multiplied by $q_{(0)k}^{-1} q_{(0)k} = 1$, and no new representation is needed.

(c) The set of diagrams which results from applying (a) and (b) to the diagrams of figure 3, represents the right-hand side of the equation of motion for H. We obtain an expansion for H itself by inverting the linear operator on the left-hand side of the equation. This corresponds to adding a broken line to the left of each diagram in figure 3. We take account of initial conditions by introducing a zero-order diagram to the expansion.

(d) Finally, carrying out the above procedure immediately yields the two second-order diagrams which represent H_2. It is tidier to anticipate this result, and substitute for the generalised diagram part H_2, when modifying diagrams (15)–(18) of figure 3.

Applying the above rules to figure 3, allows us to present the full expansion of H (to fourth order) in figure 5. Also, for completeness, the explicit forms for diagrams (15)–(18) of figure 3 can now be worked out and are given in figure 6. Thus the complete right-hand side of the energy equation is given (to fourth order) by figure 3 and figure 6.

5.3. *Expansions for the correlation and propagator functions in terms of irreducible diagrams*

The derivation of equations (4.14) and (4.16) for the exact propagator and correlation

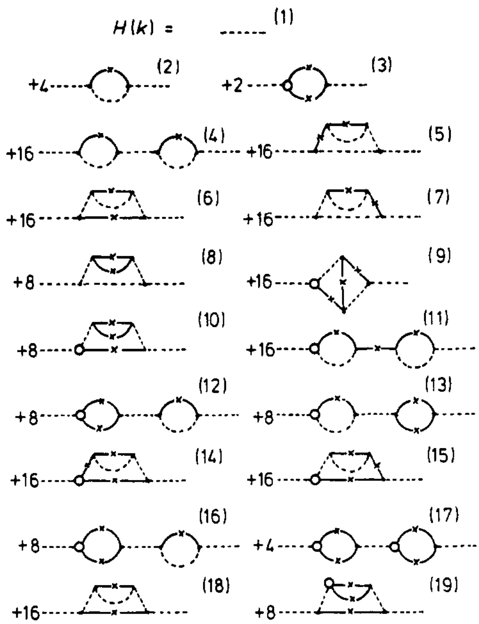

Figure 5. Expansion for the exact propagator H to fourth order.

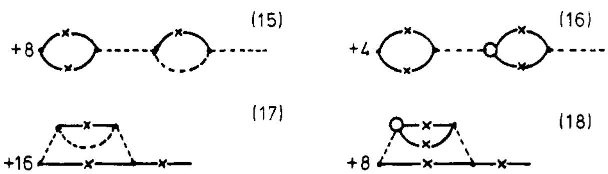

Figure 6. Diagrams (15)–(18) of figure 3, after explicit second-order diagrams have been substituted for H_2.

functions, corresponds to a summation of all those terms which add up to produce the irreducible terms, with their zero-order elements replaced by the exact forms. The resulting expansion is then truncated at second order.

This may be readily verified to fourth-order in the perturbation series. Let us introduce the following representation for the exact elements:

bold full line $\leftrightarrow$ U

bold broken line $\leftrightarrow$ H.

As before, a correlation is indicated by two full lines joined up at a 'cross'. The vertex representations (both with, and without, associated factor q_k^{-1}) are unaltered.

Considering the propagator first, and referring to figure 5, it is clear that the second-order diagrams are irreducible. Also it is readily seen that diagram (9) is the only irreducible fourth-order diagram. Thus the expansion for the exact propagator may be written as in figure 7.

Similarly, for the pair correlation, reference to figures 3 and 6 shows that the analogous expansion for the right-hand side of equation (4.16) is as given in figure 8.

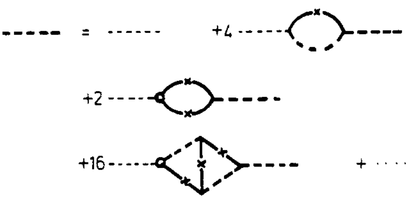

Figure 7. Diagrams corresponding to an integral equation for the exact propagator.

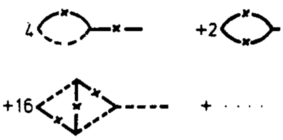

Figure 8. Irreducible diagrams corresponding to the terms on the right-hand side of equation (5.7) for the pair correlation to fourth order.

To find out which terms have been summed in our approximation procedures, we merely have to replace exact elements in the irreducible diagrams by their expansions. This will generate terms of all orders but we shall restrict our attention to fourth order. We may do this by substituting $H = H_0 + H_2$ and $Q = Q_0 + Q_2$ in each element of figures 7 and 8, as appropriate. The propagator to second order may be obtained from the first three terms on the right-hand side in figure 5 and figure 2 may be used for $Q = Q_0 + Q_2$.

Beginning with the propagator, direct substitution in the diagrams of figure 7 yields diagram numbers (1)–(5), (7)–(10) and (13)–(19) of figure 5.

A similar direct substitution of second-order expansions in figure 8 generates diagrams (1)–(4), (6)–(10) and (12)–(18) of the energy equation.

Thus our approximation omits diagrams (5) and (11) in the energy equation and diagrams (6), (11) and (12) in the equation for the propagator. Some general points may be made about this. Referring to figure 5, for the propagator, we begin by noting that although diagrams (6) and (12) are said to be omitted, they seem to be identical to (16) and (18) which are said to be included. There is in fact a hidden difference in the way these diagrams arise. Diagrams (16) and (18) have an additional factor $q^{-1}_{(0)k} q_{(0)k}$, which of course equals unity and does not appear. Thus, while we have not shown those factors which cancel to unity, in order to keep the diagrams from becoming unduly cumbersome, we have thought it proper to preserve the correct assignment of terms from the analysis.

Continuing to refer to figure 5, we are left with the problem of why three diagrams are not included. This may be connected with the related problems of double-counting (Wyld 1961) and specially symmetric diagrams (Lee 1965). In particular, if we consider diagram (11), the prototype of this diagram (in the energy equation) was originally classed by Wyld with diagrams in which the emergent line to the right was a correlation (i.e. our class II). If we were to follow this rule, then diagram (11) would have to be changed by omitting the factor q_k^{-1} and replacing the central correlation by a propagator. Then we could generate the modified diagram (11), along with the existing (12), by making the first second-order term in figure 7 more symmetric: i.e. by making the emergent left-hand propagator exact.

However, this sort of piecemeal modification would introduce difficulties elsewhere. Indeed it is the exact opposite of the step taken by Lee (1965) to avoid double-counting of certain diagrams. Thus it seems we are forced to accept this as an anomaly in our diagrammatic analysis.

6. Comparison with previous work

In § 4, we compared our time-dependent equations for the propagator and correlation functions with those resulting from direct-interaction approximation (Kraichnan 1959). We may also make a comparison with our earlier work on steady-state turbulence (McComb 1974, 1976) by assuming exponential time dependences.

Let us assume that the propagator and correlation functions may be written as:

$$H(k, t-t') = \begin{cases} \exp[-\omega_k(t-t')] & t > t' \\ 0 & t < t' \end{cases} \tag{6.1}$$

and

$$Q(k, t-t') = q_k \exp[-\omega_k(t-t')]. \tag{6.2}$$

As approximations, these forms are open to several criticisms. But they do offer a recognised prescription for making a connection between the original time-dependent theory of Kraichnan (1959) and the steady-state theory of Edwards (1964): a full discussion will be found in Leslie (1973).

Beginning with the energy equation (4.16), we simply substitute (6.1) and (6.2), put $t = t'$ and integrate over intermediate times to obtain:

$$\nu k^2 q_k = \int \frac{d^3 j\, L_{kj} q_{|k+j|}(q_j - q_k)}{\omega_k + \omega_j + \omega_{|k+j|}} \tag{6.3}$$

which is just the energy equation of Edwards (1964), and was the starting point for our previous work.

Similarly, substituting (6.1) and (6.2) into equation (4.14) for $H(k, t-t')$ yields the following equation for ω_k:

$$\omega_k = \nu k^2 + \int d^3 j \frac{L_{kj} q_{|k+j|}(q_k - q_j)}{q_k(\omega_j + \omega_{|k+j|})}. \tag{6.4}$$

This may be compared with our earlier equation for ω_k in the form (McComb 1976):

$$\omega_k = \nu k^2 + \int_{j \geqslant k} d^3 j \frac{L_{kj} q_{|k+j|}(q_k - q_j)}{q_k(\omega_k + \omega_j + \omega_{|k+j|})}.$$

These equations are very similar but two differences are immediately apparent. In (6.4) the wavenumber integration is not cut off at $j = k$ and, also unlike the earlier form, ω_k does not appear in the denominator.

That these two differences should occur together is significant. In the earlier equation, the cut-off appeared because of the antisymmetry of the integrand, under interchange of k and j. In (6.4), the absence of ω_k from the denominator, means that there is no specific antisymmetry in the integrand of (6.4) and hence there would be no justification for a cut-off.

However, equation (6.4) does possess the desirable property that the integral converges when the Kolmagoroff forms for q_k and ω_k are substituted as valid for all wavenumbers (e.g. see McComb 1976). It may readily be shown that this is due to cancellations at the singular points, $j = 0$ and $|\boldsymbol{k} + \boldsymbol{j}| = 0$. This is important as it means the theory yields the Kolmogoroff distribution as a solution.

Equations (6.3) and (6.4) could be solved together in the limiting case of infinite Reynolds number, to provide a theoretical value of the constant of proportionality in the Kolmogoroff spectrum (e.g. McComb 1976). Indeed a rough estimate suggests a value of about 1·8, which would be in better agreement with experiment than the result from the steady-state theory (McComb 1976). However, steady-state calculations of the Kolmogoroff constant are rather artificial and much more realistic tests are possible with the full time-dependent theory. In particular, we may calculate the evolution in time of the spectrum, from an arbitrary initial form. This will be the subject of further work so we shall not pursue the point here.

7. Conclusions

In this paper we have argued that the hierarchy of turbulent moment equations should be closed in terms of the relationship between eddies at different times. This is in contrast to other theories where prominence is given to the relationship between an eddy and the arbitrary stirring force. The perturbation method employed differs only slightly from that introduced by Kraichnan (1959), by the explicit (rather than implicit) presence of higher-order terms for the propagator. Thus it is not surprising that truncation at second order yields the direct-interaction energy equations. The important new feature is the presence of a diffusive-type correction term in the equation for the turbulent response.

We regard the procedures in § 4 as being the derivation of our equations. The diagrammatic analysis in § 5 was only undertaken in the spirit of establishing what was included in the theory and what was left out. It was not intended to provide a 'cut and dried' formulation. At this stage, we are forced to accept that some diagrams are apparently not included in the approximation.

It scarcely needs to be said that the graphical analysis does not justify our approximation procedure in terms of its accuracy. We do not know the magnitude of those fourth-order terms which are excluded from the second-order renormalised terms. And, even if all reducible fourth-order diagrams had been accounted for, truncation at second order would still raise questions about the magnitude of fourth- and higher-order irreducible terms. In the present state of knowledge, the sheer complexity of the higher orders makes such questions imponderable. Thus the theory must still be regarded as rather tentative and we have the, not unfamiliar, situation that the final equations have to be judged by the results to which they lead.

There is also the question of realisability. Early methods (e.g. cumulant discard; quasi-normality: see Leslie 1973) led to unphysical results like negative energy spectra. It has been argued by Kraichnan that the existence of a stochastic model representation, for which the direct-interaction approximation is exact, guarantees the realisability of the latter (and indeed that of some other theories: Leslie 1973). We do not have such a model representation for the present theory thus it must be emphasised that equations (4.14) and (4.16) may not lead to realisable forms in all circumstances. This situation is, of course, not unique. Previous theories which yield the $-\frac{5}{3}$ solution (Kraichnan 1965, Edwards and McComb 1969) also lack a model representation.

It is to be hoped that we will be able to answer some of these questions from a numerical study of the time-dependent equations. This will be the subject of future work. For the present, we would suggest that it is encouraging that the equations are realisable for the particular case of the Kolmogoroff spectrum. And, although in view of intermittency effects there is currently some doubt about the underlying validity of the Kolmogoroff distribution, we have pointed out previously (McComb 1974) that we take the pragmatic view that the $-\frac{5}{3}$ law (or something very close to it) has been found experimentally. Thus whatever the true inertial-range form may turn out to be, the Kolmogoroff solution may be regarded as a reasonable practical test for any turbulence theory. Moreover, it is the rigorous infinite-Reynolds number solution for any closure (like the present one) in which intermittency effects are suppressed.

Finally, we should not ignore the fact that our theory manages to suppress the divergence in the response integral within a purely Eulerian framework. It has previously been argued by Kraichnan (1965) that a divergent response integral reflects a spurious interaction between the energy-containing and inertial ranges of wavenumber. He has shown that this interaction may be removed by requiring that his equations should be invariant under random Galilean transformations (Kraichnan 1965, 1971). Although this requirement is evidently of considerable physical significance and importance, it is not clear to me that it represents a *fundamental* requirement in the way the deterministic transformation does. Therefore it should perhaps be emphasised that the present theory does not invoke such a prescription nor do we claim that our equations are invariant to stochastic Galilean transformations. Possibly the ability to yield the Kolmogoroff distribution as a solution without satisfying this particular invariance requirement may be one of the more interesting features of this work.

References

Balescu R and Senatorski A 1970 *Ann. Phys., NY* **58** 587
Edwards S F 1964 *J. Fluid Mech.* **18** 239
Edwards S F and McComb W D 1969 *J. Phys. A: Gen. Phys.* **2** 157
Herring J R 1965 *Phys. Fluids* **8** 2219
Hopf E 1952 *Archs Ration. Mech. Analysis* **1** 87
Kraichnan R H 1959 *J. Fluid Mech.* **5** 497
—— 1965 *Phys. Fluids* **8** 575
—— 1971 *J. Fluid Mech.* **47** 513
Lee L L 1965 *Ann. Phys., NY* **32** 292
Leslie D C 1973 *Developments in the Theory of Turbulence* (London: Oxford University Press)
McComb W D 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 632
—— 1976 *J. Phys. A: Math. Gen.* **9** 179
Nakano T 1972 *Ann. Phys., NY* **73** 326
Phythian R 1969 *J. Phys. A: Gen. Phys.* **2** 181
Wyld H W 1961 *Ann. Phys., NY* **14** 143